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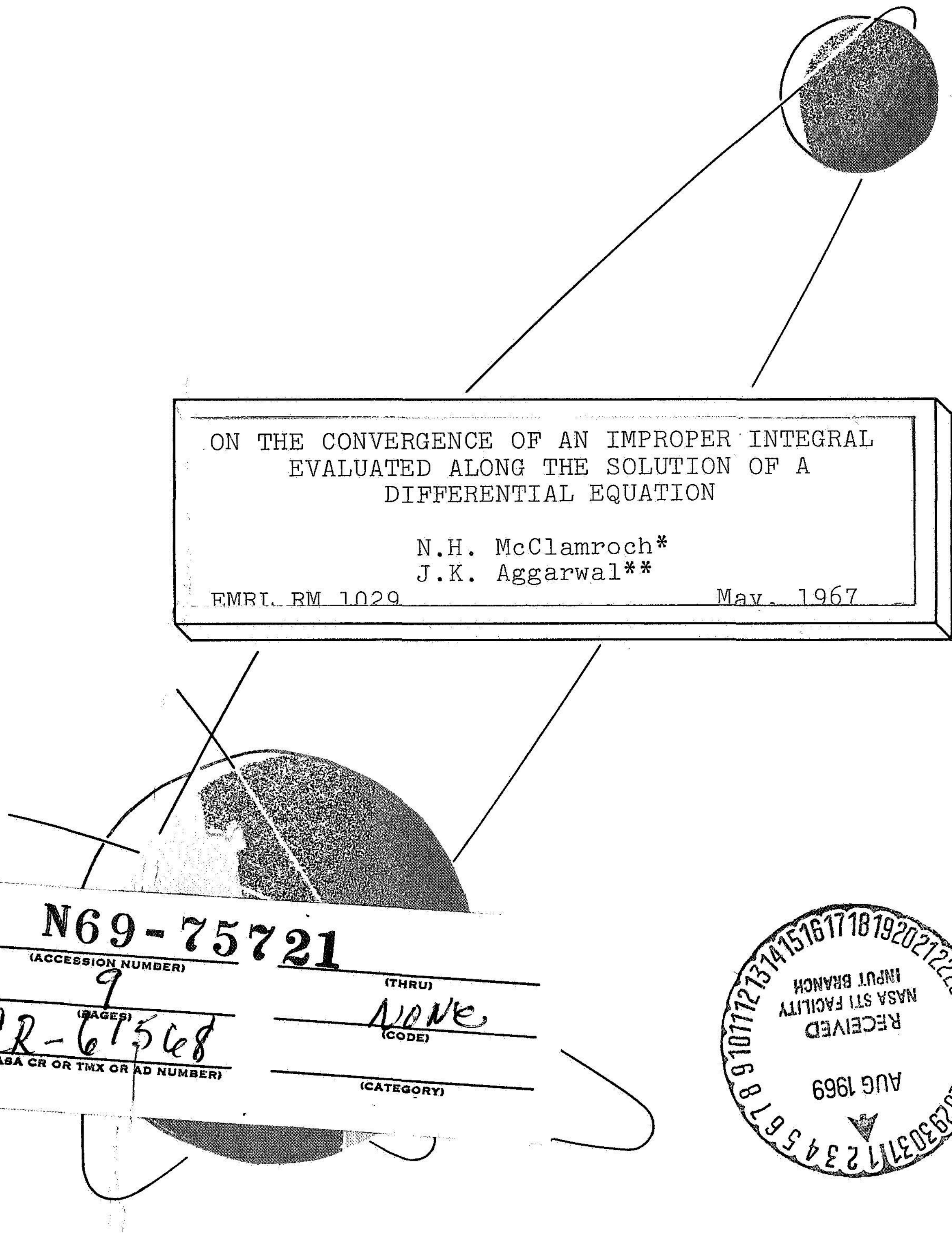

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Introduction

The study of the convergence (existence) and the evaluation of improper integrals has received much attention. In this paper, we shall consider conditions under which the convergence of a particular class of improper integrals can be guaranteed. In particular, the integrand is not specified in closed form but is evaluated in terms of the unique solution to an associated differential equation. The improper integral can be written as

$$\int_{t_o}^{\infty} g(\phi(t,x_o,t_o),t)dt \tag{1}$$

where g is a specified scalar function of the n-vector ϕ and the time t. The function ϕ represents the solution to the vector differential equation

$$\frac{dx}{dt} \equiv \dot{x} = f(x,t) \tag{2}$$

with the initial condition

$$x(t_o) = x_o. \tag{3}$$

$f(x,t)$ is a function with values in the Euclidean space R^n which is defined on some set $S \times I = \{(x,t) \varepsilon R^n \times R \mid ||x||<r, t>\theta\}$. It is assumed that $f(x,t)$ is sufficiently smooth on $S \times I$ such that for any $x_o \varepsilon S$ and any $t_o \varepsilon I$ there exists for all $t \geqslant t_o$ a unique solution in S. The possibility of a finite escape time in (θ,∞) is excluded. Also $x = 0$ is the trivial solution of

(2) so that $f(0,t) = 0$ on I. It is assumed that the scalar valued function g in (1) is continuous on S x I and $g(0,t) = 0$ on I.

As a motivation to this study, it has been conjectured that a necessary and sufficient condition for the convergence of (1) for an arbitrary $x_o \varepsilon S$ and $t_o \varepsilon I$ is that (2) be asymptotically stable in S x I. That this is not a sufficient condition for the convergence of (1) is readily observed by considering the following example. Let (1) and (2) be given by

$$\int_0^\infty \phi^2(t)\, dt \qquad (4)$$

and

$$\dot{x} = -x^3 \quad , \quad x(0) = x_o, \qquad (5)$$

where $\phi(t)$ is the solution to (5). The trivial solution of (5) is uniformly asymptotically stable in the large; however, it is observed that the integral (4) does not converge for any nonzero x_o.

Our goal in this work is to obtain sufficient conditions for the convergence of (1) under rather weak conditions. Several results guaranteeing this convergence are presented and proved. The relations of the theorems to the stability properties of (2) are discussed. Finally we consider the use of (1) as a measure of the performance of (2).

Results on Convergence

In what follows, the main results of the paper are presented.

Theorem: Let $g(\phi,t)$ be non-negative on $S \times I$. Let $\phi(t,x_o,t_o)\to 0$ as $t\to\infty$ for any $x_o \varepsilon S$ and any $t_o \varepsilon I$. Let $V(x,t)$ be a non-negative decrescent scalar valued function with continuous first partial derivatives on $S \times I$, and $V(0,t) = 0$ on I. If there exists a $\rho>0$ such that

$$\dot{V}(x,t) + \rho g(x,t) \leqslant 0 \text{ on } S \times I, \tag{6}$$

then (1) converges for any $x_o \varepsilon S$ and any $t_o \varepsilon I$.

Proof: $\dot{V}(x,t) + \rho g(x,t) \leqslant 0$ on $S \times I$. Let $x \equiv \phi(t,x_o,t_o)$ for some $x_o \varepsilon S$ and $t_o \varepsilon I$. Then for all $t \geqslant t_o$,

$$\dot{V}(\phi(t),\ t) + \rho g(\phi(t),\ t) \leqslant 0.$$

Integrating over the interval $[t_o,T]$ we obtain

$$V(\phi(T),\ T) - V(x_o,t_o) + \rho \int_{t_o}^{T} g(\phi(t),\ t)dt \leqslant 0.$$

Since $T\to\infty$ implies that $\phi(T)\to 0$ and $V(x,t)$ is continuous and decrescent, then $T\to\infty$ implies $V(\phi(T),\ T)\to 0$. In other words, for any specified $\varepsilon \geqslant 0$, there is sufficiently large T', depending on ε, x_o, and t_o, such that for all $T>T'$

$$\int_{t_o}^{T} g(\phi(t),\ t)dt \leqslant \frac{1}{\rho} V(x_o,t_o) - \frac{\varepsilon}{\rho}.$$

Thus, for any $T > t_o$

$$\int_{t_o}^{T} g(\phi(t),\ t)dt \leqslant \frac{1}{\rho}\, V(x_o,t_o).$$

Since $g(x,t)$ is non-negative, $\int_{t_o}^{T} g(\phi(t),\ t)dt$ is obviously monotonic with T. Thus, we assert that

$$\int_{t_o}^{T} g(\phi(t),\ t)dt$$

converges as $T\to\infty$. Q.E.D.

From the Theorem, we can obtain the following corollary.

<u>Corollary 1.</u> Let $g(\phi,t)$ be positive definite on $R^n xI$. Let $V(x,t)$ be a positive definite decrescent scalar function with continuous first partial derivatives on $R^n xI$. Also $V(0,t) = 0$ on I, and $V(x,t)\to\infty$ as $||x||\to\infty$ uniformly on I. If there exists a $\rho > o$ such that

$$\dot{V}(x,t) + \rho g(x,t) \leqslant 0 \text{ on } R^n \text{ x } I \tag{7}$$

then (1) converges for any $x_o \varepsilon R^n$ and any $t_o \varepsilon I$.

Proof: Consider $V(x,t)$ as a Liapunov function. From (7), $\dot{V}(x,t)$ is obviously negative definite, so that $x = 0$ is uniformly asymptotically stable in the large and $\phi(t,x_o,t_o)\to 0$ as $t\to\infty$ [1]. Thus the conditions of the theorem are satisfied with $S = R^n$. Q.E.D.

If f and g are not explicit functions of time, then (1) and (2) are given by

$$\int_{o}^{\infty} g(\phi(t))dt \tag{8}$$

and

$$\dot{x} = f(x) \quad , \quad x(0) = x_o, \tag{9}$$

where, without loss of generality, we have taken $t_o=0$. As before, $\phi(t)$ denotes the unique solution of (9). For this case the following corollary is obtained.

Corollary 2. Let $g(\phi)$ be non-negative. Let $V(x)$ be a positive definite scalar valued function with continuous first partial derivatives and $V(0) = 0$. Define Ω_ℓ as the region where $V(x) < \ell$, and assume that Ω_ℓ is bounded. If there exists a $\rho > 0$ such that

$$\dot{V}(x) + \rho g(x) \leqslant 0 \text{ on } \Omega_\ell \tag{10}$$

and the only invariant set [3] contained in $\Omega_\ell \cap \{x | \dot{V}(x) + \rho g(x) = 0\}$ is the trivial solution $x = 0$, then the integral in (8) converges for any $x_o \varepsilon \Omega_\ell$.

Proof: Consider $V(x)$ as a Liapunov function. From (10) and the hypothesis, it follows, using a result of LaSalle [3], that $x = 0$ is asymptotically stable in Ω_ℓ and $\phi(t,x_o,t_o) \to 0$ as $t \to \infty$. Thus, the conditions of the Theorem are satisfied with $S = \Omega_\ell$. Q.E.D.

Although Corollaries 1 and 2 are less general than the Theorem, they are more useful in applications since one does not have to check that $\phi(t) \to 0$ as $t \to \infty$.

Discussion

We return to the conjecture that asymptotic stability of the origin of the differential equation (2) is necessary and sufficient condition for the convergence of the improper integral (1). Asymptotic stability is not a necessary condition for the convergence of (1) since asymptotic stability is not implied by the hypothesis of the Theorem; only quasi-asymptotic stability [1] is implied. Referring to the example, asymptotic stability is obviously not sufficient. Thus the conjecture is incorrect in both respects.

These results have implications in several areas. Not only have conditions for the convergence of (1) been given, but an upper bound on the integral has been established. This upper bound has been used by McClamroch and Aggarwal [4] to deduce the sensitivity characteristics of (1) with respect to certain types of functional changes in the differential equation (2).

These results also have application in asymptotic control theory. Here one is interested in determining a control u within some class U such that a performance measure given by (1) is minimized and the differential equation

$$\dot{x} = f(x,t,u) \tag{11}$$

is satisfied. Unless the convergence of (1) for some

$u \varepsilon U$ can be guaranteed the use of such a performance measure is not justified. This idea has been considered previously by Bellman and Bucy [2] for the case where (11) is linear.

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